



TITLE:

On Pfaffian Systems on  $\mathbb{P}_2(\mathbb{C})$  with Logarithmic Singularities (複素領域における微分方程式)

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On Pfaffian systems on  $\mathbb{P}_2(\mathbb{C})$   
with logarithmic singularities

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0. Introduction.

Let  $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$  be an algebraic subset of  $\mathbb{P}_2(\mathbb{C})$  where for each  $i$ ,  $\mathcal{A}_i$  is irreducible and given by an irreducible polynomial equation in the homogeneous coordinates on  $\mathbb{P}_2(\mathbb{C})$ :

$$P_i(x_1, x_2, x_3) = 0.$$

We are considering Pfaffian systems of the form

$$(1) \quad dz = \omega z$$

$$\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$$

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\* J.S.P.S. visiting Professor July 1978 to the end of October 1978.

where the  $A_i$ 's are constant square matrices of order  $m$ .

In this lecture we are going to speak about

1) the set  $\mathcal{R}(1)$  of relations between the  $A_i$ 's implied by the condition

$$d\omega = \omega \wedge \omega = 0$$

2) the relations in  $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$

3) the nature of the solutions of (1)

4) the Riemann-Hilbert problem.

In this lecture there are no deep results but only simple remarks.

# 1. Some examples.

1.1.  $\mathcal{A} = \emptyset$ ,  $\omega = 0$ ,  $\pi_1(\mathbb{P}_2(\mathbb{C})) = 1$  and  $\mathbb{C}^m$  is the vector space of solutions.

1.2.  $\mathcal{A} = \bigcup_{i=1}^3 \mathcal{A}_i$ ,  $\mathcal{A}_i = \{x \in \mathbb{P}_2(\mathbb{C}) \mid x_i = 0\}$

$$(1.2) \quad \omega = \sum_{i=1}^3 A_i \frac{dx_i}{x_i}.$$

Then  $\mathcal{R}(1)$  :

$$\begin{cases} [A_i, A_j] = 0 & i \neq j \\ A_1 + A_2 + A_3 = 0. \end{cases}$$

And

$\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$  is abelian.

A fundamental matrix of solutions is given by:

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

and all solutions are elementary functions. The Riemann-Hilbert problem can easily be solved with a system of the form (1.2).

1.3. The Pfaffian system associated to the hypergeometric functions in two variables.

The hypergeometric function  $F_1$  is given by the system of partial differential equations

$$x(1-x)(x-y) \frac{\partial^2 z}{\partial x^2} + [\gamma(x-y) - (\alpha+\beta+1)x^2 + (\alpha+\beta-\beta'+1)xy + \beta'y] \frac{\partial z}{\partial x}$$

$$- \beta y(1-y) \frac{\partial z}{\partial y} - \alpha \beta (x-y)z = 0$$

$$y(1-y)(y-x) \frac{\partial^2 z}{\partial y^2} + [\gamma(y-x) - (\alpha+\beta'+1)y^2 + (\alpha+\beta'-\beta+1)xy + \beta x] \frac{\partial z}{\partial y}$$

$$- \beta' x(1-x) \frac{\partial z}{\partial x} - \alpha \beta' (y-x)z = 0$$

$$(x-y) \frac{\partial^2 z}{\partial x \partial y} - \beta \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 0.$$

But the map

$$z \longmapsto \begin{pmatrix} z \\ x \frac{\partial z}{\partial x} \\ y \frac{\partial z}{\partial y} \end{pmatrix}$$

transforms this "complicated" system into the following one of type (1) which is completely integrable.

$$dz = \omega z$$

$$\omega = \sum_{i=1}^3 A_i \frac{dx_i}{x_i} + \sum_{i=1}^3 B_i \frac{du_i}{u_i}$$

$$u_i = x_j - x_k \quad j \neq k, j \neq i, k \neq i,$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha\beta' & -\beta' & \gamma-\alpha-\beta'-1 \end{pmatrix}$$

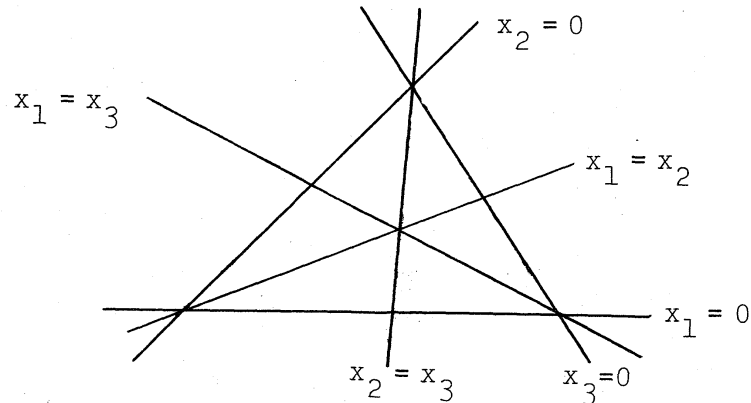
$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha\beta & \gamma-\alpha-\beta-1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & -1 & -1 \\ \alpha\beta & \alpha+\beta & \beta \\ \alpha\beta' & \beta' & \alpha+\beta' \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{pmatrix}$$

and we have  $d\omega = \omega \wedge \omega = 0$ . The singular set:



The same thing can be done for the functions  $F_2$  and  $F_3$  by using the map

$$z \longmapsto \begin{pmatrix} z \\ x \frac{\partial z}{\partial x} \\ y \frac{\partial z}{\partial y} \\ xy \frac{\partial^2 z}{\partial x \partial y} \end{pmatrix}$$

for  $F_2$  the singular set is

$$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\} \cup \\ \{x_1 + x_2 - x_3 = 0\} \cup \{x_3 = 0\},$$

for  $F_3$  the singular set is

$$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\} \cup \\ \{x_2 x_3 - x_1 x_3 - x_1 x_2 = 0\}.$$

We don't know if a transformation does exist for  $F_4$ . If it does exist it will be more complicated than for  $F_1, F_2, F_3$ .

1.4. Let  $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$  an arbitrary algebraic subset of  $\mathbb{P}_2(\mathbb{C})$ .

Then let  $\{A_i\}_{i=1,2,\dots,n}$  a set of permutable matrices satisfying

$$\sum_{i=1}^n \deg(P_i) A_i = 0.$$

Then the system

$$dz = \omega z$$

with

$$\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$$

is completely integrable and has  $\mathcal{A}$  as singular set. But we cannot call this system "generic".

Problem I. For any algebraic set  $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$  in  $\mathbb{P}_2(\mathbb{C})$  does there exist a completely integrable Pfaffian system of type (1) with singularities on  $\mathcal{A}$  and which is not example 1.4. ?

For some singular sets the answer is yes: examples  $F_1$ ,  $F_2$ ,  $F_3$ .

In example 1.4, we have a global fundamental matrix of solutions

$$\prod_{i=1}^n P_i^{A_i}$$

which is given by elementary functions. Then we can reformulate problem I in

Problem I'. Find the algebraic sets  $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$  in  $\mathbb{P}_2(\mathbb{C})$

for which there exists a completely integrable Pfaffian system having singularities on  $\mathcal{A}$  and which does not have a global fundamental matrix of solutions which is elementary.

2. The condition  $d\omega = \omega \wedge \omega = 0$ .

Let  $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$ ,  $\mathcal{A}_i : P_i(x_1, x_2, x_3) = 0$  (irreducible).

2.1. The  $\mathcal{A}_i$ 's are normal crossing.

Consider  $\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$  then it is easy to see that

$$d\omega = \omega \wedge \omega \iff [A_i, A_j] = 0 \text{ for all } i, j \text{ } i \neq j.$$

And  $\omega$  is well defined on  $\mathbb{P}_2(\mathbb{C})$  if we add the condition

$$\sum_{i=1}^n (\deg P_i) A_i = 0.$$

And in this case problems I and I' are solved and the solutions are all elementary functions.

The Riemann-Hilbert problem can be solved with a system of type (1). In fact  $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$  is abelian, then any linear representation

$$\chi : \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A}) \longrightarrow GL(m, \mathbb{C})$$

is abelian.

The fundamental group has  $n$ -generators  $g_1, g_2, \dots, g_n$  related by

$$\prod_{i=1}^n g_i^{\deg P_i} = 1.$$

Set  $\chi(g_i) = D_i$  and  $\tilde{A}_i = \frac{1}{2\pi i} \log D_i$ . Then

$$[\tilde{A}_i, \tilde{A}_j] = 0$$

and

$$\sum_{i=1}^n (\deg P_i) \tilde{A}_i = 2m\pi i I$$

and by choosing in a suitable way the determination of  $\log D_i$ , it is possible to find the set of  $A_i$ 's satisfying

$$[A_i, A_j] = 0$$



and

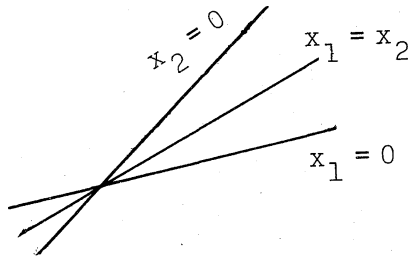
$$\sum_{i=1}^n (\deg P_i) A_i = 0.$$

2.2. The  $A_i$ 's are not normal crossing.

2.2.1. An example.

$$A = \bigcup_{i=1}^3 A_i, \quad A_1 = \{x_1 = 0\}, \quad A_2 = \{x_2 = 0\}$$

$$A_3 = \{x_1 = x_2\}.$$



$$\text{Set } \omega = A_1 \frac{dx_1}{x_1} + A_2 \frac{dx_2}{x_2} + A_3 \frac{d(x_1 - x_2)}{x_1 - x_2}.$$

And  $d\omega = \omega \wedge \omega$  gives

$$[A_1, A_1 + A_2 + A_3] = 0$$

$$[A_2, A_1 + A_2 + A_3] = 0$$

$$[A_3, A_1 + A_2 + A_3] = 0$$

which are trivial consequence of

$$A_1 + A_2 + A_3 = 0$$

which has to be satisfied to have a system on  $\mathbb{P}_2(\mathbb{C})$ .

This means that any set of matrices  $A_1, A_2, A_3$  such that  $A_1 + A_2 + A_3 = 0$  solves the problem I, the solutions are not elementary but nearly related the hypergeometric Gauss function of one variable.

2.2.2. Let us consider a Pfaffian form

$$\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$$

$A_i : P_i = 0$ ,  $P_i$  irreducible. Then by a finite number of monoidal transformations (blowing up), we construct a manifold  $X$  with an analytic map

$$\sigma : X \longrightarrow \mathbb{P}_2(\mathbb{C}).$$

There exists in  $X$ , a divisor

$$A^* = \bigcup_{j=1}^m A_j^* \quad (m > n)$$

where the  $A_j^*$  are normal crossing and for each  $i \in [1, 2, \dots, n]$  there exists  $j(i) \in [1, 2, \dots, m]$  such that

$$\sigma : A_{j(i)}^* \xrightarrow{\sim} A_i.$$

But  $A^*$  contains some exceptional divisor coming from blowing up of the singularities of  $A$ . Moreover  $\sigma$  is an isomorphism from

$$X - A^* \text{ onto } \mathbb{P}_2(\mathbb{C}) - A.$$

This implies that

$$\pi_1(X - A^*) \simeq \pi_1(\mathbb{P}_2(\mathbb{C}) - A).$$

Denote by  $\sigma^*(\omega)$  the inverse image of  $\omega$  by  $\sigma$ , then

$\sigma^*(\omega)$  has logarithmic poles on  $A^*$ . Then we have for each  $i \in [1, 2, \dots, n]$

$$\text{R\'es}_{A_i}(\sigma^*\omega) = \text{R\'es}_{A_i}(\omega) = A_i$$

and for each exceptional divisor  $B$

$$\text{R\'es}_B(\sigma^*\omega) = \sum_{i=1}^n r_i \text{R\'es}_{A_i}(\omega)$$

where the  $r_i$ 's are integers.

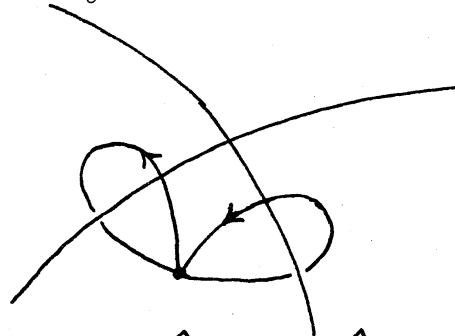
By local computations in  $X$ , we can easily prove the following

$$\omega \wedge \omega = 0 \iff \sigma^*\omega \wedge \sigma^*\omega = 0 \iff [\text{R\'es}_{A_i}^{\sigma^*}(\omega), \text{R\'es}_{A_j}^{\sigma^*}(\omega)] = 0$$

for all  $i, j$   $i \neq j$  with  $A_i^* \cap A_j^* \neq \emptyset$ .

This result can also be seen by using the following remark.

Locally in  $X$  near a point  $M \in A_i^* \cap A_j^*$ , choose a simple path  $\gamma_i$  surrounding  $A_i^*$  and a path  $\gamma_j$  surrounding  $A_j^*$  having the same origin and such that



$$\int_{\gamma_i} \frac{dP_i^*}{P_i^*} = 2\pi\sqrt{-1}$$

$$\int_{\gamma_j} \frac{dP_j^*}{P_j^*} = 2\pi\sqrt{-1}.$$

Denote by  $\hat{\gamma}_i$  and  $\hat{\gamma}_j$  the local homotopy classes of  $\gamma_i$  and  $\gamma_j$ . We have

$$\hat{\gamma}_i \hat{\gamma}_j = \hat{\gamma}_j \hat{\gamma}_i$$

(the local fundamental group is abelian). This means that

there exists a two cell  $S_{ij}$  homotopic to zero such that

$$\partial S_{ij} = \gamma_i \gamma_j \gamma_i^{-1} \gamma_j^{-1}$$

and

Proposition 1.

$$\int_{S_{ij}} \sigma^*(\omega) \wedge \sigma^*(\omega) = +4\pi^2 [\text{R\'es}_{A_j^*} \sigma^*(\omega), \text{R\'es}_{A_i^*} \sigma^*(\omega)]$$

and as a corollary

$$\omega \wedge \omega = 0 \implies [\text{R\'es}_{A_j^*} \sigma^*(\omega), \text{R\'es}_{A_i^*} \sigma^*(\omega)] = 0$$

for all  $i, j$   $i \neq j$ .

Let us summarize the results:

The complete integrability condition is equivalent to the commutation of the r\'esidues of  $\sigma^*(\omega)$  for  $A_i^*$  and  $A_j^*$  when  $A_i^* \cap A_j^* \neq \emptyset$ .

This means that for  $\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$  we have

$$\omega \wedge \omega = 0 \iff \begin{cases} \sum_{i=1}^n (\deg P_i) A_i = 0 \\ [\text{R\'es}_{A_i^*} \sigma^*(\omega), \text{R\'es}_{A_j^*} \sigma^*(\omega)] = 0 \\ \text{where} \\ \text{R\'es}_{A_j^*} \sigma^*(\omega) = \text{R\'es}_{A_j} \omega \end{cases}$$

where  $A_j$  is not an exceptional divisor and

$$\text{R\'es}_{A_j^*} \sigma^*(\omega) = \sum_{i=1}^n r_i^j \text{R\'es}_{A_i} \omega$$

where  $r_i^j$  are integers.

### 2.3. The group of the relation $\omega \wedge \omega = 0$ .

Let us construct a group  $G$  in the following way.

To each irreducible component  $A_i$  of  $A$  is associated in an abstract way a generator  $g_i$  of  $G$ . Then to each relation among the  $A_i$ 's we associate a relation between the  $g_i$ 's in the following way.

$$\text{To } \sum_{i=1}^n P_i A_i = 0 \quad P_i = \deg P_i \quad \text{let us associate}$$

$$g_1^{P_1} g_2^{P_2} \dots g_n^{P_n} = 1 \quad \text{and} \quad \tau(g_1^{P_1} g_2^{P_2} \dots g_n^{P_n}) = 1$$

for all circular permutation  $\tau$  of the factors.

$$\text{To } [\text{Rés}_{A_i} \sigma^*(\omega), \text{Rés}_{A_j} \sigma^*(\omega)] = 0$$

$$[g_1^{r_1^i} g_2^{r_2^i} \dots g_n^{r_n^i}, g_1^{r_1^j} \dots g_n^{r_n^j}] = 1$$

and

$$[\tau(g_1^{r_1^i} \dots g_n^{r_n^i}), \tau(g_1^{r_1^j} \dots g_n^{r_n^j})] = 1$$

for each circular permutation  $\tau$  of the factors.

#### Examples.

1. See 1.2.

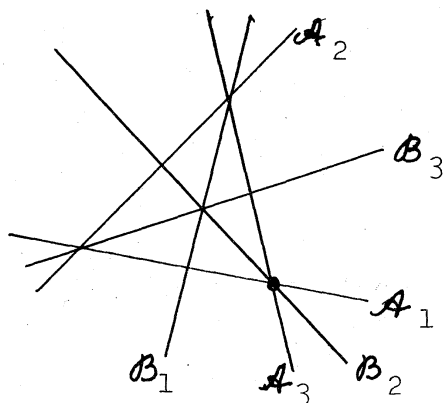
$$G = \{g_1, g_2, g_3; [g_i, g_j] = 1, g_1 g_2 g_3 = 1, \tau(g_1 g_2 g_3) = 1\}$$

and  $G$  is a free abelian with two generators.

2. See 2.1. Normal crossing case.

$G$  is a free abelian with  $(n-1)$ -generators.

3. See 1.3. Hypergeometric function  $F_1$ .



The group  $G$  has 6 generators

$$a_1, a_2, a_3, b_1, b_2, b_3$$

corresponding to  $A_1, A_2,$

$A_3, B_1, B_2, B_3$ .

The relations are up to cyclic

permutations as indicated above:

$$[a_1, b_1] = [a_2, b_2] = [a_3, b_3] = 1$$

$$[a_1, b_2 a_3] = [a_1, b_3 a_2] = 1$$

$$[a_2, b_3 a_1] = [a_2, b_1 a_3] = 1$$

$$[a_3, b_1 a_2] = [a_3, b_2 a_1] = 1$$

$$[b_1, b_2 b_3] = [b_1, a_2 a_3] = 1$$

$$[b_2, b_1 b_3] = [b_2, a_1 a_3] = 1$$

$$[b_3, b_1 b_2] = [b_3, a_1 a_2] = 1$$

and

$$a_1 a_2 a_3 b_1 b_2 b_3 = 1.$$

4. See 2.2.1.

The group  $G$  has three generators  $g_1, g_2, g_3$  and up to

cyclic permutation as indicated we have the relations

$$[g_i, g_1 g_2 g_3] = 1 \quad i = 1, 2, 3.$$

but also  $g_1 g_2 g_3 = 1$ . As a consequence  $G$  is a free group with two generators.

### 3. Some problems.

I. If problem I in section 1 is solvable for an algebraic set  $\mathcal{A}$ , what are the relations between

- 1) the monodromy of the Pfaffian system and the group  $G$
- 2)  $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$  and  $G$  ?

II. In which cases is the Riemann-Hilbert problem solvable by a Pfaffian system of type (1) ? Or such that  $\sigma^*(\omega)$  is of type (1) in each coordinate system in  $X$  ?

Remark: In many particular examples when  $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$  is known we have  $G = \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ .

To finish this summary let me thank Professor M. Oka of the University of Tokyo with whom I have had a very interesting discussion about fundamental groups of the complementary of an algebraic curve in  $\mathbb{P}_2(\mathbb{C})$ .